Fractional systems and fractional Bogoliubov hierarchy equations

Vasily E. Tarasov*

Skobeltsyn Institute of Nuclear Physics, Moscow State University, Moscow 119992, Russia

We consider the fractional generalizations of the phase volume, volume element, and Poisson brackets. These generalizations lead us to the fractional analog of the phase space. We consider systems on this fractional phase space and fractional analogs of the Hamilton equations. The fractional generalization of the average value is suggested. The fractional analogs of the Bogoliubov hierarchy equations are derived from the fractional Liouville equation. We define the fractional reduced distribution functions. The fractional analogs of the Vlasov equation and the Debye radius are considered.

I. INTRODUCTION

Derivatives and integrals of fractional order have found many applications in recent studies of scaling phenomena [1–6]. In Ref. [7], coordinate fractional derivatives in the Fokker-Planck equation were used. It is known that Fokker-Planck equation can be derived from the Liouville equation [8]. The Liouville equation is derived from the normalization condition and the Hamilton equations [9]. In the Hamilton equation $dq_k/dt=p_k/m$, $dp_k/dt=F_k(q,p)$ we have only the time derivatives. The usual normalization condition leads to the usual (nonfractional) Liouville equation. If we would like to derive the fractional Liouville equation then we must use a fractional normalization condition. In Ref. [10], the fractional Liouville equation was derived from fractional normalization condition.

The natural question arises: What could be the physical meaning of the fractional normalization condition? This physical meaning can be the following: the system is to be found somewhere in the fractional phase space. The fractional normalization condition can be considered as a normalization condition for the distribution function in a fractional phase space. In order to use this interpretation we must define a fractional phase space. The first interpretation of the fractional phase space is connected with fractional dimension space. The fractional dimension interpretation follows from the formulas for dimensional regularizations. This interpretation was suggested in Ref. [10]. In this paper we consider the second interpretation of the fractional phase space. This interpretation follows from the fractional measure of phase space [10] that is used in the fractional integrals. The fractional phase space is considered as a phase space that is described by the fractional powers of coordinates and momenta (q_k^α, p_k^α). In this case, the fractional normalization condition for the distribution function and the fractional average values are considered as a condition and values for the fractional space. In general, these systems are non-Hamiltonian dissipative systems for the usual phase space (q,p).

It is known that Bogoliubov equations can be derived from the Liouville equation and the definition of the average

value [11–14]. In Ref. [10], the fractional Liouville equation is derived from the fractional normalization condition. In this paper we define the fractional analog of the average value and reduced distributions. Then we derive the fractional Bogoliubov equations from the fractional Liouville equation and the definition of the fractional average value. We derive the fractional analog of the Vlasov equation and the Debye radius.

In Ref. [10], the Riemann-Liouville definition of the fractional integration and differentiation is used. Therefore in this paper we use this definition of fractional integration and differentiation.

In Sec. II, we define the fractional phase space volume. The fractional phase volume element for the fractional phase space is considered. We define the fractional analog of the Poisson bracket. In Sec. III, we consider the fractional systems. We discuss the free motion of the fractional systems, the fractional harmonic oscillator and fractional analog of the Hamiltonian systems. In Sec. IV, the fractional average values and some notations are considered. We define the reduced one-particle and two-particle distribution functions. In Sec. V, the fractional Liouville equation for n-particle system is written. We derive the first fractional Bogoliubov equation from the fractional average value and the fractional Liouville equation. The second fractional Bogoliubov equation is considered. In Sec. VI, we derive the fractional analog of the Vlasov equation and the Debye radius for fractional systems. Finally, a short conclusion is given in Sec. VII.

II. FRACTIONAL PHASE VOLUME AND POISSON BRACKETS

A. Fractional phase volume for configuration space

Let us consider the phase volume for the region such that $x \in [a;b]$. The usual phase volume is defined by

$$V_1 = \int_a^b dx = \int_a^y dx + \int_y^b dx. \qquad (1)$$

The fractional integrations are defined [16] by

$$I_{a+}^\alpha 1 = \frac{1}{\Gamma(\alpha)} \int_a^y \frac{dx}{(y-x)^{1-\alpha}},$$

$$I_{b-}^{\alpha}1 = \frac{1}{\Gamma(\alpha)} \int_y^b \frac{dx}{(x-y)^{1-\alpha}}.$$

Using these notations, we get Eq. (1) in the form

$$V_1 = I_{a+}^1 1 + I_{b-}^1 1. \tag{2}$$

The fractional generalization of this phase volume can be defined by

$$V_\alpha = I_{a+}^\alpha 1 + I_{b-}^\alpha 1. \tag{3}$$

The fractional phase volume integral (3) can be written in the form

$$V_\alpha = \frac{1+\lambda^\alpha}{\Gamma(\alpha)} \int_0^{b-y} \xi^{\alpha-1} \, d\xi, \tag{4}$$

where λ is defined by

$$\lambda = \frac{y-a}{b-y}.$$

In order to have the symmetric limits of the phase volume integral we can use the following equation:

$$V_\alpha = \frac{(1+\lambda^\alpha)}{2} \int_{-(b-y)}^{+(b-y)} d\mu_\alpha(x), \tag{5}$$

where

$$d\mu_\alpha(x) = \frac{|x|^{\alpha-1} \, dx}{\Gamma(\alpha)} = \frac{dx^\alpha}{\alpha \Gamma(\alpha)}. \tag{6}$$

Here we use the following notation for fractional power of coordinates:

$$x^\alpha = \beta(x)(x)^\alpha = \text{sgn}(x)|x|^\alpha, \tag{7}$$

where $\beta(x) = [\text{sgn}(x)]^{\alpha-1}$. The function $\text{sgn}(x)$ is equal to $+1$ for $x \geq 0$, and this function is equal -1 for $x < 0$.

B. Fractional phase volume for phase space

Using Eq. (5), we have the phase volume for the two-dimensional phase space in the form

$$V_\alpha = \frac{(1+\lambda_q^\alpha)}{2} \frac{(1+\lambda_p^\alpha)}{2} \int_{-(q_b-y')}^{(q_b-y)} \int_{-(p_b-y')}^{(p_b-y)} \frac{dq^\alpha \wedge dp^\alpha}{[\alpha\Gamma(\alpha)]^2}, \tag{8}$$

where

$$dq^\alpha \wedge dp^\alpha = \alpha^2 |qp|^{\alpha-1} \, dq \wedge dp. \tag{9}$$

The fractional measure for the region B of $2n$-dimensional phase space can be defined by the equation

$$\mu_\alpha(B) = V_\alpha = \int_B g(\alpha) d\mu_\alpha(q,p), \tag{10}$$

where $d\mu_\alpha(q,p)$ is a phase volume element,

$$d\mu_\alpha(q,p) = \prod_{k=1}^n \frac{dq_k^\alpha \wedge dp_k^\alpha}{[\alpha\Gamma(\alpha)]^2}, \tag{11}$$

and $g(\alpha)$ is a numerical multiplier,

$$g(\alpha) = \frac{1}{4^n} \prod_{k=1}^n g_k(\alpha).$$

If the domain B of the phase space is defined by $q_k \in \mathbb{R}^1$ and $p_k \in \mathbb{R}^1$, then $g_k(\alpha) = 1$. If this domain is defined by $q_k \in [q_{ak}; q_{bk}]$ and $p_k \in [p_{ak}; p_{bk}]$, then

$$g_k(\alpha) = \left[1 + \left(\frac{q_{bk}-y_k}{y_k-q_{ak}}\right)^\alpha\right]\left[1 + \left(\frac{p_{bk}-y_k'}{y_k'-p_{ak}}\right)^\alpha\right]. \tag{12}$$

It is easy to see that the fractional measure depends on the fractional powers (q^α, p^α).

C. Fractional exterior derivatives

In Eq. (8), we use the usual exterior derivative

$$d = \sum_{k=1}^{2n} dx_k \frac{\partial}{\partial x_k}. \tag{13}$$

Obviously, this derivative can be represented in the form

$$d = \sum_{k=1}^{2n} dx_k^\alpha \frac{\partial}{\partial x_k^\alpha}, \tag{14}$$

where x^α is defined by Eq. (7).

Note that the volume element of fractional phase space can be realized by fractional exterior derivatives [15],

$$d^\alpha = \sum_{k=1}^n dq_k^\alpha \frac{\partial^\alpha}{[\partial(q_k-y_k)]^\alpha} + \sum_{k=1}^n dp_k^\alpha \frac{\partial^\alpha}{[\partial(p_k-y_k')]^\alpha},$$

in the following form:

$$dq^\alpha \wedge dp^\alpha = \left(\frac{4}{\Gamma^2(2-\alpha)} + \frac{1}{\Gamma^2(1-\alpha)}\right)^{-1} (qp)^{\alpha-1} d^\alpha q \wedge d^\alpha p.$$

D. Fractional Poisson brackets

We can define the fractional generalization of the Poisson brackets in the form

$$\{A,B\}_{(\alpha)} = \sum_{k=1}^n \left(\frac{\partial^\alpha A}{(\partial q_k)^\alpha} \frac{\partial^\alpha B}{(\partial p_k)^\alpha} - \frac{\partial^\alpha A}{(\partial p_k)^\alpha} \frac{\partial^\alpha B}{(\partial q_k)^\alpha}\right), \tag{15}$$

where we use the fractional derivatives [16]. It is known [16] that the derivative of a constant need not be zero,

$$\frac{\partial^\alpha 1}{(\partial x)^\alpha} = \frac{1}{\Gamma(1-\alpha)} x^{-\alpha}. \tag{16}$$

This equation leads us to the correlation between coordinates q and momenta p in the form

$$\frac{\partial^\alpha p}{(\partial q)^\alpha} = \frac{pq^{-\alpha}}{\Gamma(1-\alpha)}, \quad \frac{\partial^\alpha q}{(\partial p)^\alpha} = \frac{qp^{-\alpha}}{\Gamma(1-\alpha)}. \tag{17}$$

It is easy to see that q and p are not independent variables in the usual sense. Therefore the fractional analog (15) of the Poisson brackets is not convenient.

In the general case ($n \geqslant 2$), the Poisson brackets $\{q_k, q_l\}_{(\alpha)}$ and $\{p_k, p_l\}_{(\alpha)}$ (with $k \neq l$) are not equal to zero. If we consider $n = 2$, then we have

$$\{q_1, q_2\}_{(\alpha)} = -\gamma_2(\alpha)q_1 q_2[(q_1 p_1)^{-\alpha} - (q_2 p_2)^{-\alpha}], \quad (18)$$

where the coefficient $\gamma_2(\alpha)$ is defined by the equation

$$\gamma_2(\alpha) = \frac{1}{\Gamma^2(1-\alpha)} - \frac{2}{\Gamma(1-\alpha)\Gamma(2-\alpha)}. \quad (19)$$

Moreover, the Poisson brackets with the unit are not equal to zero. Using Eq. (16), we get

$$\{1, q\}_{(\alpha)} = \gamma_2(\alpha)q^{1-\alpha}p^{\alpha}, \quad (20)$$

$$\{1, p\}_{(\alpha)} = -\gamma_2(\alpha)q^{-\alpha}p^{1-\alpha}. \quad (21)$$

Therefore the fractional Poisson brackets (15) are not convenient.

We can use the fractional power Poisson brackets:

$$\{A, B\}_\alpha = \sum_{k=1}^{n} \left(\frac{\partial A}{\partial q_k^\alpha} \frac{\partial B}{\partial p_k^\alpha} - \frac{\partial A}{\partial p_k^\alpha} \frac{\partial B}{\partial q_k^\alpha} \right), \quad (22)$$

where we use notations (7). Obviously, the relations

$$\{q_k^\alpha, q_l^\alpha\}_\alpha = 0, \quad \{p_k^\alpha, p_l^\alpha\}_\alpha = 0, \quad \{q_k^\alpha, p_l^\alpha\}_\alpha = \delta_{kl},$$

are realized for these Poisson brackets. Therefore the fractional Poisson brackets (22) are more convenient.

E. Fractal dimension of space

The interpretation of the fractional phase space is connected with the fractional measure of phase space. The parameter α defines the space with the fractional phase measure (10) and (11). It is easy to prove that the velocity of the fractional phase volume change is defined by the equation

$$\frac{dV_\alpha}{dt} = \int_B \Omega_\alpha(q, p, t)g(\alpha)d\mu_\alpha(q, p),$$

where

$$\Omega_\alpha = \left\{ \frac{dq_t^\alpha}{dt}, p_t^\alpha \right\}_\alpha + \left\{ q_t^\alpha, \frac{dp_t^\alpha}{dt} \right\}_\alpha. \quad (23)$$

Equation (23) is proved in Ref. [10]. The form of the omega function allows us to consider a different class of the systems that are described by the fractional powers of coordinates and momenta.

The interpretation of the fractional phase space can be derived from the fractional measure that is used in the fractional integrals. The interpretation of the fractional phase space can be connected with fractional dimension. We have two arguments for this point of view.

(i) Let us use the well-known formulas for dimensional regularizations [17]:

$$\int \rho(x)d^n x = \frac{2\pi^{n/2}}{\Gamma(n/2)} \int_0^\infty \rho(x)x^{n-1} dx. \quad (24)$$

Using Eq. (24), we get that the fractional integral can be considered [10] as an integral for the fractional dimension space

$$\frac{\Gamma(\alpha/2)}{2\pi^{\alpha/2}\Gamma(\alpha)} \int \tilde{\rho}(x,t)d^\alpha x \quad (25)$$

up to the numerical factor $\Gamma(\alpha/2)/[2\pi^{\alpha/2}\Gamma(\alpha)]$.

(ii) Let us consider the well-known definition of the fractal mass dimension. The equations that define the fractal dimensions have the passage to the limit. This passage makes difficult the practical application to the real fractal media. The other dimensions, which can be easily calculated from the experimental data, are used in the empirical investigations. For example, the mass fractional dimension [18,19] can be easily measured.

The properties of the fractal media like mass obeys a power law relation,

$$M(r) = kr^{D_m}, \quad (26)$$

where M is the mass of fractal medium, r is a box size (or a sphere radius), and D_m is a mass fractal dimension. The amount of mass of a medium inside a box of size r has a power law relation (26).

The fractal dimension can be calculated by the box counting method which means drawing a box of size r and counting the mass inside. To calculate the mass fractal dimension, take the logarithm of both sides of Eq. (26):

$$\ln(M) = D_m \ln(r) + \ln k.$$

The log-log plot of M and r gives us the slope D_m, the fractal dimension. When we graph $\ln(M)$ as a function of $\ln(r)$, we get a value of about D_m which is the fractal dimension of fractal media.

The power law relation (26) can be naturally derived by using the fractional integral. In order to describe the fractal media, we suggest to use the space with fractional measure.

Let us consider the line distribution of the mass. If we consider the mass of the homogeneous distribution ($\rho = \text{const}$) in the ball region W with radius r, then we have

$$M_1(r) = \int_{-r}^{+r} \rho(x)dx = 2\rho \int_0^r dx = 2r^1. \quad (27)$$

In this case, $D_m = 1$. Let us consider line mass distribution in the fractional space. In that case, a ball of radius r covers a mass

$$M_\alpha(r) = \frac{2\rho r^\alpha}{\alpha\Gamma(\alpha)}.$$

This equation can be proved by the fractional generalization of Eq. (27) in the form

$$M_\alpha(r) = \frac{2}{\Gamma(\alpha)} \int_0^r \rho(\xi)\xi^{\alpha-1}\,d\xi = \rho V_\alpha(r) = \frac{2\rho r^\alpha}{\alpha\Gamma(\alpha)}. \quad (28)$$

The initial points in the fractional integral (4) are set to zero, and $a=-r$, $b=r$. The fractal dimension of particle system and fractal media is defined as the exponent of r in the growth law for mass $M(r)$ or number of particles,

$$n(r) = M(r)/m = (k/m)r^{D_m}. \quad (29)$$

Here m is a particle mass. Thus we see that the fractal dimension D_m of particle system (in the fractional space) is α, i.e., $D_m=\alpha$. Therefore the space with fractional measure (4) can be considered as a space with fractal dimension $D_m=\alpha$.

As the result the space with fractional measure (4) can be used to describe the particle systems and medium with non-integer mass dimension.

III. FRACTIONAL SYSTEMS

A. Equations of motion

In Sec. II we prove that the fractional measure (11) depends on the fractional powers (q^α,p^α), and Poisson brackets (22) with fractional powers are more convenient. Therefore we can consider a different class of mechanical systems that are described by the fractional powers of coordinates and momenta. We can consider the fractional power of the coordinates as a convenient way to describe systems in the fractional dimension space.

Let us consider a classical system with the mass M. Suppose this system is described by the dimensional coordinates $\bar{q}_k$ and momenta $\bar{p}_k$ that satisfy the following equations of motion:

$$\frac{d\bar{q}_k}{d\bar{t}} = \frac{\bar{p}_k}{M}, \quad \frac{d\bar{q}_k}{d\bar{t}} = f_k(\bar{q},\bar{p},\bar{t}). \quad (30)$$

Let q_0 be the characteristic scale in the configuration space; p_0 be the characteristic momentum, F_0 be the characteristic value of the force, and t_0 be the typical time. Let us introduce the *dimensionless variables*

$$q_k = \frac{\bar{q}_k}{q_0}, \quad p_k = \frac{\bar{p}_k}{p_0}, \quad t = \frac{\bar{t}}{t_0}, \quad F_k = \frac{f_k}{F_0}.$$

Here and later we use q_k and p_k as dimensionless variables. Using Eq. (30) for dimensional physical variables $(\bar{q},\bar{p})$, we get the equations for dimensionless variables,

$$\frac{dq_k}{dt} = \frac{p_k}{m}, \quad \frac{dq_k}{d\bar{t}} = AF_k(q,p,t), \quad (31)$$

where $m=Mq_0/t_0 p_0$ is the dimensionless mass, and

$$A = \frac{t_0 F_0}{p_0}. \quad (32)$$

Using the dimensionless variables (q,p,t), we can consider the fractional generalization of Eq. (31) in the form

$$\frac{dq_k^\alpha}{dt} = \frac{p_k^\alpha}{m}, \quad \frac{dq_k^\alpha}{d\bar{t}} = AF_k(q^\alpha,p^\alpha,t), \quad (33)$$

where we use the following notations:

$$q_k^\alpha = \beta(q)(q_k)^\alpha = \text{sgn}(q_k)|q_k|^\alpha, \quad (34)$$

$$p_k^\alpha = \beta(p)(p_k)^\alpha = \text{sgn}(p_k)|p_k|^\alpha. \quad (35)$$

Here $k=1,\dots,n$, and $\beta(x)$ is defined by Eq. (7).

A system is called a fractional system if the phase space of the system can be described by the fractional powers of coordinates (34) and momenta (35).

The fractional phase space can be considered as a phase space for the fractional systems. This interpretation follows from the fractional measure that is used in the fractional integrals.

We can consider the fractional systems in the usual phase space (q,p) and in the fractional phase space (q^α,p^α). In the second case, the equations of motion for the fractional systems have more simple form. Therefore we use the fractional phase space. The fractional space is considered as a space with the fractional measure that is used in the fractional integrals.

The fractional generalization of the conservative Hamiltonian system is described by the equation

$$\frac{dq_k^\alpha}{dt} = \frac{\partial H}{\partial p_k^\alpha}, \quad \frac{dp_k^\alpha}{dt} = -\frac{\partial H}{\partial q_k^\alpha}, \quad (36)$$

where H is a fractional analog of the Hamiltonian. Note that the function H is the invariant of the motion. Using the fractional Poisson brackets (22), we have

$$\frac{dq_k^\alpha}{dt} = \{q_k^\alpha,H\}_\alpha, \quad \frac{dp_k^\alpha}{dt} = \{p_k^\alpha,H\}_\alpha. \quad (37)$$

Here we use Poisson brackets (22). These equations describe the system in the fractional phase space (q^α,p^α). For the usual phase space (q,p), the fractional Hamiltonian systems are described by the equations

$$\frac{dq_k}{dt} = \frac{(q_k p_k)^{1-\alpha}}{\alpha^2}\frac{\partial H}{\partial p_k}, \quad \frac{dp_k}{dt} = -\frac{(q_k p_k)^{1-\alpha}}{\alpha^2}\frac{\partial H}{\partial q_k}. \quad (38)$$

The fractional Hamiltonian systems are non-Hamiltonian systems in the usual phase space (q,p). A classical system (in the usual phase space) is called Hamiltonian if the right-hand sides of the equations

$$\frac{dq_k}{dt} = g_k(q,p), \quad \frac{dp_k}{dt} = f_k(q,p), \quad (39)$$

satisfy the following Helmholtz conditions [20]:

$$\frac{\partial g_k}{\partial p_l} - \frac{\partial g_l}{\partial p_k} = 0, \quad \frac{\partial g_k}{\partial q_l} - \frac{\partial f_l}{\partial p_k} = 0, \quad \frac{\partial f_k}{\partial q_l} - \frac{\partial f_l}{\partial q_k} = 0. \quad (40)$$

It is easy to prove that the Helmholtz conditions are not satisfied. Therefore the fractional Hamiltonian system (38) is a non-Hamiltonian system in the usual phase space (q,p). The fractional phase space allows us to write the equations of

motion for the fractional systems in the simple form (36).

If we have $dq_k^\alpha/dt = p_k^\beta/m$, then the fractional analog of the Hamiltonian can be considered in the form

$$H = \sum_{k,l=1}^n \frac{\alpha p_k^{\alpha+\beta}}{m(\alpha+\beta)} + U(q). \tag{41}$$

If $\beta = \alpha$, then we have the fractional analog of the kinetic energy $T = p^{2\alpha}/2m$.

The omega function for the system (39) in the usual phase space (q,p) is defined by the equation

$$\Omega = \sum_{k=1}^n \left(\frac{\partial g_k}{\partial q_k} + \frac{\partial f_k}{\partial p_k} \right). \tag{42}$$

If the omega function is negative $\Omega < 0$, then the system is called a dissipative system. If $\Omega \neq 0$, then the system is a generalized dissipative system. For the fractional Hamiltonian systems (38), the omega function (42) is not equal to zero. Therefore the fractional Hamiltonian systems are the general dissipative systems in the usual phase space.

It is known that the non-Hamiltonian and dissipative systems (for example, $F = -\gamma p$) are not invariant under the Galilean transformation. In general, the fractional systems are non-Hamiltonian systems for the usual phase space (q,p). Therefore the Galilean transformations for the equations of motion are not considered in this paper. The fractional analogs of the Hamiltonian systems can be invariant under the fractional generalization of the Galilean transformation.

It is not hard to prove that the fractional systems (41) are connected with the non-Gaussian statistics. Classical dissipative and non-Hamiltonian systems can have the canonical Gibbs distribution as a solution of the stationary Liouville equations [21]. Using the methods [21], it is easy to prove that some of fractional dissipative systems can have fractional Gibbs distribution (non-Gaussian statistic),

$$\rho(q,p) = \exp[\mathcal{F} - H(q,p)]/kT, \tag{43}$$

as a solution of the fractional Liouville equations [10]. The interest in and relevance of fractional kinetic equations is a natural consequence of the realization of the importance of non-Gaussian statistics of many dynamical systems. There is already a substantial literature studying such equations in one or more space dimensions.

B. Free motion of fractional system

Let us consider the free motion of the fractional system that is defined by the following equations:

$$\frac{dq^\alpha}{dt} = \frac{p^\alpha}{m}, \quad \frac{dp^\alpha}{dt} = 0. \tag{44}$$

The solutions of these equations with the conditions $q(0) = q_1$ and $p(0) = p_1$ have the form

$$q^\alpha(t) = m^{-1}p_1^\alpha t + q_1^\alpha, \quad p(t) = p_1. \tag{45}$$

We can conclude that the free motion of the fractional system is described by

$$q(t) \sim (t - t_1)^{1/\alpha}, \tag{46}$$

where the parameter t_1 is defined by q_1.

Let us consider the special cases of the parameter α: 0.5, 1/3, 0.6.

(i) If the parameter α is equal to 0.5, then we have the solution of Eq. (44) in the form

$$q(t) = z(q_1,p_1,t)(at^2 + bt + c), \tag{47}$$

where we use the following notations:

$$z(q_1,p_1,t) = \text{sgn}(\sqrt{|p_1|}t - m\sqrt{|q_1|}), \tag{48}$$

$$a = |p_1|/m, \quad b = 2\sqrt{|q_1 p_1|}/m, \quad c = |q_1|. \tag{49}$$

(ii) If the parameter α is equal to 1/3, then we have the solution of Eq. (44) in the form

$$q(t) = \frac{p_1}{m^3}t^3 + \frac{3p_1^{2/3}q_1^{1/3}}{m^2}t^2 + \frac{3p_1^{1/3}q_1^{2/3}}{m}t + q_1. \tag{50}$$

(iii) If the parameter α is equal to 0.6, then we have the solution of Eq. (44) in the form

$$q(t) = \left(\frac{p_1^{3/5}}{m}t + q_1^{3/5} \right)^{5/3}. \tag{51}$$

If $q_1 = 0$, then $q(t) \sim t^{5/3}$.

For the usual phase space (q,p), the equations of motion for the fractional system can be represented in the following form (39). For the free fractional system, we have

$$\frac{dq_k}{dt} = \frac{q^{1-\alpha}p^\alpha}{\alpha m}, \quad \frac{dp_k}{dt} = 0. \tag{52}$$

The omega function for the free fractional system is equal to the following function:

$$\Omega = \frac{1-\alpha}{\alpha m}q^{-\alpha}p^\alpha. \tag{53}$$

In general, this function does not equal to zero and the phase volume of the usual phase space changes. Using Eq. (45), we get

$$\Omega = \frac{1-\alpha}{\alpha}(t + t_1)^{-1}, \tag{54}$$

where $t_1 = mq_1^\alpha/p_1^\alpha$. If $q_1 = 0$, then the omega function is proportional to $1/t$. Therefore the velocity of elementary phase volume change for the free motion in the usual phase space (q,p) is inversely proportional to the time. We suppose that the initial momentum is not equal to zero $p_1 \neq 0$.

For the fractional phase space (q^α,p^α), we define [10] the omega function Ω_α, in Eq. (23). This "fractional" omega function is equal to zero for the free motion of fractional system. Therefore the fractional phase space (q^α,p^α) is more convenient than usual phase space (q,p).

C. Fractional harmonic oscillator

Let us consider the fractional harmonic oscillator, which is defined in Ref. [10] by the equations

$$\frac{dq^{\alpha}}{dt} = \frac{p^{\alpha}}{m}, \quad \frac{dp^{\alpha}}{dt} = -m\omega^2 q^{\alpha}, \tag{55}$$

where ω is a dimensionless variable. The solutions of these equations of motion have the form

$$q^{\alpha}(t) = a\cos(\omega t + \varphi), \quad p^{\alpha}(t) = -m\omega a\sin(\omega t + \varphi), \tag{56}$$

where parameters a, φ are defined by

$$a = \sqrt{q_1^{2\alpha} + \frac{p_1^{2\alpha}}{\omega^2 m^2}}, \quad tg\varphi = -\frac{p_1^{\alpha}}{m\omega q_1^{\alpha}}. \tag{57}$$

If $\alpha = 0.5$, then we have

$$q(t) = \text{sgn}[\cos(\omega t + \varphi_1)]a^2\cos^2(\omega t + \varphi). \tag{58}$$

If $\alpha = 1/3$, then we have $q(t) = a^3\cos^3(\omega t + \varphi)$. The fractional harmonic oscillator has the following integral of motion:

$$H = \frac{p^{2\alpha}}{2m} + \frac{m\omega^2 q^{2\alpha}}{2} = \text{const.} \tag{59}$$

This function can be considered as a fractional analog of Hamiltonian for Eq. (36).

For the fractional harmonic oscillator the function Ω_{α} is equal to zero in the fractional phase space (q^{α}, p^{α}). Therefore this system is a conservative Hamiltonian nondissipative system in the fractional space. *If we use the usual phase space to describe the fractional harmonic oscillator, then this system is conservative non-Hamiltonian dissipative system.* Note that the conservative non-Hamiltonian systems are considered in Ref. [22].

In the usual phase space (q, p) the equations of motion for fractional oscillator have the form

$$\frac{dq}{dt} = \frac{1}{\alpha m}q^{1-\alpha}p^{\alpha}, \quad \frac{dp}{dt} = -\frac{m\omega^2}{\alpha}q^{\alpha}p^{1-\alpha}. \tag{60}$$

The omega function for the usual phase space (q, p) is defined by Eq. (42) in the form

$$\Omega = \frac{1-\alpha}{\alpha m}q^{-\alpha}p^{-\alpha}(p^{2\alpha} - m^2\omega^2 q^{2\alpha}). \tag{61}$$

The elementary phase volume of the usual phase space changes. The velocity of this change is equal to the omega function. Substituting the solution (56) into Eq. (61), we have

$$\Omega = \frac{1-\alpha}{\alpha}2\omega\cot 2(\omega t + \varphi), \tag{62}$$

where we use $\sin^2\beta - \cos^2\beta = -\cos 2\beta$ and $\cot\beta = \cos\beta/\sin\beta$. Therefore the fractional harmonic oscillator is a general dissipative system in the usual phase space (q, p).

For the fractional phase space (q^{α}, p^{α}), the fractional harmonic oscillator is conservative nondissipative Hamiltonian system. Therefore the fractional phase space (q^{α}, p^{α}) is more convenient than usual phase space (q, p).

The question arises: What are the fundamentals (different from Hamilton principle), which can lead to the system of

dynamic equations (60)? For the fractal media the harmonic oscillator is defined by Eq. (55). For the usual phase space (q, p), this equation has form (60). Therefore the system (60) can be considered as a system,

$$\frac{dq}{dt} = \frac{p}{M}, \quad \frac{dp}{dt} = -M(\omega/\alpha)^2 q, \tag{63}$$

where the dimensionless mass $M(q, p) = m\alpha(p/q)^{1-\alpha}$ satisfies the scaling relation

$$M(\lambda_1 q, \lambda_2 p) = (\lambda_2/\lambda_1)^{1-\alpha}M(q, p).$$

This property can be formulated in the following form. If we consider the scale transformation of the characteristic values in the form

$$q_0 \to q_0/\lambda_1, \quad p_0 \to p_0/\lambda_2,$$

then we have transformation of the mass

$$M \to (\lambda_2/\lambda_1)^{1-\alpha}M.$$

In the general case, this scaling law can be described by the renormalization group approach [23].

Note that the system (63) is a non-Hamiltonian system. We consider the fractional phase space form (55) of Eq. (60) as more fundamental. The fractional harmonic oscillator is an oscillator in the fractional phase space that can be considered as a fractal medium. Therefore the fractional oscillator can be interpreted as an elementary excitation of some fractal medium with noninteger mass dimension.

D. Curved phase space

The fractional system with the fractional analog (41) of the Hamiltonian can be considered as a nonlinear system with

$$H(q^{\alpha}, p^{\alpha}) = \sum_{k=1}^{n}\frac{1}{2}g_{kl}(q, p)p_k p_l + U(q). \tag{64}$$

Note that this fractional Hamiltonian (64) defines a nonlinear one-dimensional sigma model [24,25] in the curved phase space with metric $g_{kl}(q, p) = m^{-1}p_k^{\alpha+\beta-2}\delta_{kl}$. This means that we use the curved phase space. Note that this fractional Hamiltonian is used in equations of motion (38) that define the non-Hamiltonian flow in the usual phase space.

The curved phase space is used in the Tuckerman approach to the non-Hamiltonian statistical mechanics [26–34]. In their approach the suggested invariant phase space measure of non-Hamiltonian systems is connected with the metric of the curved phase space. This metric defines the generalization of the Poisson brackets. The generalized (non-Hamiltonian) bracket is suggested in Refs. [31,32]. For these brackets, the Jacobi relations will not be satisfied. This requires the application of non-Lie algebras (in which the Jacobi identity does not hold) and analytic quasigroups (which are nonassociative generalizations of groups). In the paper [35], we show that the analogues of Lie algebras and groups for non-Hamiltonian systems are Valya algebras (anticommutative algebras whose commutants are Lie subalgebras)

and analytic commutant-associative loops [whose commutants are associative subloops (groups)]. Unfortunately, non-Lie algebras and its representations have not been thoroughly studied. Nevertheless, the Riemannian treatment of the phase space is very interesting. This approach allows us to consider the connection between the fractal dimensional phase space and non-Lie algebras of the vector fields in the space.

IV. FRACTIONAL AVERAGE VALUES AND REDUCED DISTRIBUTIONS

A. Fractional average values for configuration space

Let us derive the fractional generalization of the equation that defines the average value of the classical observable $A(q,p)$.

The usual average value for the configuration space

$$\langle A \rangle_1 = \int_{-\infty}^{\infty} A(x)\rho(x)dx \tag{65}$$

can be written in the form

$$\langle A \rangle_1 = \int_{-\infty}^{y} A(x)\rho(x)dx + \int_{y}^{\infty} A(x)\rho(x)dx. \tag{66}$$

Using the notations

$$(I_+^{\alpha}f)(y) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{y} \frac{f(x)dx}{(y-x)^{1-\alpha}},$$

$$(I_-^{\alpha}f)(y) = \frac{1}{\Gamma(\alpha)} \int_{y}^{\infty} \frac{f(x)dx}{(x-y)^{1-\alpha}},$$

average value (66) can be rewritten in the form

$$\langle A \rangle_1 = (I_+^1 A\rho)(y) + (I_-^1 A\rho)(y).$$

The fractional generalization of this equation is defined by

$$\langle A \rangle_\alpha = (I_+^{\alpha} A\rho)(y) + (I_-^{\alpha} A\rho)(y). \tag{67}$$

We can rewrite Eq. (67) in the form

$$\langle A \rangle_\alpha = \frac{1}{2} \int_{-\infty}^{\infty} [(A\rho)(y-x) + (A\rho)(y+x)]d\mu_\alpha(x), \tag{68}$$

where we use

$$d\mu_\alpha(x) = \frac{|x|^{\alpha-1} dx}{\Gamma(\alpha)} = \frac{dx^\alpha}{\alpha\Gamma(\alpha)}, \tag{69}$$

and x^α is defined by Eq. (7).

B. Fractional average values for phase space

Let us introduce some notations to consider the fractional average value for phase space.

(i) Let us define the tilde operators T_{x_k} and $T[k]$. The operator T_{x_k} is defined by

$$T_{x_k} f(\ldots, x_k, \ldots) = \frac{1}{2}(f(\ldots, x_k' - x_k, \ldots) + f(\ldots, x_k' + x_k, \ldots)).$$

This operator allows us to rewrite the functions

$$\frac{1}{4}[A(q'-q,p'-p,t)\rho(q'-q,p'-p,t)$$

$$+ A(q'+q,p'-p,t)\rho(q'+q,p'-p,t)$$

$$+ A(q'-q,p'+p,t)\rho(q'-q,p'+p,t)$$

$$+ A(q'+q,p'+p,t)\rho(q'+q,p'+p,t)]$$

in the simple form

$$T_q T_p(A(q,p,t)\rho(q,p,t)).$$

Let us consider k particle that is described by generalized coordinates q_{ks} and generalized momenta p_{ks}, where $s = 1, \ldots, m$. The operator $T[k]$ is defined by the relation $T[k] = T_{q_{k1}} T_{p_{k1}} \cdots T_{q_{km}} T_{p_{km}}$. For the n-particle system phase space, we use the operator $T[1, \ldots, n] = T[1] \cdots T[n]$.

(ii) Let us define the integral operators $\hat{I}_{x_k}^\alpha$ and $\hat{I}^\alpha[k]$. The operator $\hat{I}_{x_k}^\alpha$ is defined by the equation

$$\hat{I}_{x_k}^\alpha f(x_k) = \int_{-\infty}^{+\infty} f(x_k)d\mu_\alpha(x_k). \tag{70}$$

In this case, fractional integral (68), which defines the average value, can be rewritten in the form

$$\langle A \rangle_\alpha = \hat{I}_x^\alpha T_x A(x)\rho(x).$$

Let us define the phase space integral operator $\hat{I}^\alpha[k]$ for k particle by $\hat{I}^\alpha[k] = \hat{I}_{q_{k1}}^\alpha \hat{I}_{p_{k1}}^\alpha \cdots \hat{I}_{q_{km}}^\alpha \hat{I}_{p_{km}}^\alpha$, i.e., we use

$$\hat{I}^\alpha[k] f(\mathbf{q}_k, \mathbf{p}_k) = \int f(\mathbf{q}_k, \mathbf{p}_k)d\mu_\alpha(\mathbf{q}_k, \mathbf{p}_k). \tag{71}$$

Here $d\mu_\alpha(\mathbf{q}_k, \mathbf{p}_k)$ is an elementary $2m$-dimensional phase volume that is defined by the equation

$$d\mu_\alpha(\mathbf{q}_k, \mathbf{p}_k) = [\alpha\Gamma(\alpha)]^{-2m} dq_{k1}^\alpha \wedge dp_{k1}^\alpha \wedge \cdots \wedge dq_{km}^\alpha \wedge dp_{km}^\alpha.$$

For the n-particle system phase space, we use the integral operator $\hat{I}^\alpha[1, \ldots, n] = \hat{I}^\alpha[1] \cdots \hat{I}^\alpha[n]$.

(iii) Let us define the fractional analog of the average values A for the phase space for n-particle system. Using the suggested notations, we can define the fractional average value by the relation

$$\langle A \rangle_\alpha = \hat{I}^\alpha[1, \ldots, n] T[1, \ldots, n] A\rho_n. \tag{72}$$

In the simple case ($n = m = 1$), the fractional average value is defined by the equation

$$\langle A \rangle_\alpha = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\mu_\alpha(q,p) T_q T_p A(q,p)\rho(q,p). \tag{73}$$

Note that the fractional normalization condition [10] is a special case of this definition of the average value

$$\langle 1 \rangle_\alpha = 1.$$

C. Reduced distribution functions

In order to derive a fractional analog of the Bogoliubov hierarchy equations we must define the reduced distributions.

Let us consider a classical system with fixed number n of identical particles. Suppose k particle is described by the *dimensionless* generalized coordinates q_{ks} and generalized momenta p_{ks}, where $s = 1, \ldots, m$. We use the following notations $\mathbf{q}_k = (q_{k1}, \ldots, q_{km})$ and $\mathbf{p}_k = (p_{k1}, \ldots, p_{km})$.

The state of this system can be described by *dimensionless* n-particle distribution function ρ_n in the $2mn$-dimensional phase space,

$$\rho_n(\mathbf{q}, \mathbf{p}, t) = \rho(\mathbf{q}_1, \mathbf{p}_1, \ldots, \mathbf{q}_n, \mathbf{p}_n, t).$$

We assume that distribution function is invariant under the permutations of identical particles [36]:

$$\rho(\ldots, \mathbf{q}_k, \mathbf{p}_k, \ldots, \mathbf{q}_l, \mathbf{p}_l, \ldots, t) = \rho(\ldots, \mathbf{q}_l, \mathbf{p}_l, \ldots, \mathbf{q}_k, \mathbf{p}_k, \ldots, t).$$

In this case, the average values for the classical observables can be simplified. We use the tilde distribution functions

$$\widetilde{\rho}_n(\mathbf{q}, \mathbf{p}, t) = T[1, \ldots, n] \rho_n(\mathbf{q}, \mathbf{p}, t), \tag{74}$$

and the function $\widetilde{\rho}_1$ that is defined by the equation

$$\widetilde{\rho}_1(\mathbf{q}, \mathbf{p}, t) = \widetilde{\rho}(\mathbf{q}_1, \mathbf{p}_1, t) = \hat{I}^\alpha[2, \ldots, n] \widetilde{\rho}_n(\mathbf{q}, \mathbf{p}, t). \tag{75}$$

This function is called one-particle reduced distribution function. The function is defined for the $2m$-dimensional phase space. Obviously, that one-particle distribution function satisfies the normalization condition [10]

$$\hat{I}^\alpha[1] \widetilde{\rho}_1(\mathbf{q}, \mathbf{p}, t) = 1. \tag{76}$$

Two-particle reduced distribution function $\widetilde{\rho}_2$ is defined by the fractional integration of the n-particle distribution function over all $\mathbf{q}_k$ and $\mathbf{p}_k$, except $k = 1, 2$:

$$\widetilde{\rho}_2(\mathbf{q}, \mathbf{p}, t) = \widetilde{\rho}(\mathbf{q}_1, \mathbf{p}_1, \mathbf{q}_2, \mathbf{p}_2, t) = \hat{I}^\alpha[3, \ldots, n] \widetilde{\rho}_n(\mathbf{q}, \mathbf{p}, t). \tag{77}$$

V. FRACTIONAL LIOUVILLE AND BOGOLIUBOV EQUATIONS

A. Fractional Liouville equation

The fractional generalization of the Liouville equation is derived in Ref. [10]. Let us consider the Hamilton equations for n-particle system in the form

$$\frac{dq_{ks}^\alpha}{dt} = G_s^k(q^\alpha, p^\alpha), \quad \frac{dp_{ks}^\alpha}{dt} = A F_s^k(q^\alpha, p^\alpha, t). \tag{78}$$

The evolution of n-particle distribution function ρ_n is described by the Liouville equation. The fractional Liouville equation [10] for n-particle distribution function has the form

$$\frac{d\widetilde{\rho}_n}{dt} + \Omega_\alpha \widetilde{\rho}_n = 0. \tag{79}$$

This equation can be derived [10] from the fractional normalization condition

$$\hat{I}^\alpha[1, \ldots, n] \widetilde{\rho}_n(\mathbf{q}, \mathbf{p}, t) = 1. \tag{80}$$

In the Liouville equation d/dt is a total time derivative,

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \sum_{k,s=1}^{n,m} \frac{dq_{ks}}{dt} \frac{\partial}{\partial q_{ks}} + \sum_{k,s=1}^{n,m} \frac{dp_{ks}}{dt} \frac{\partial}{\partial p_{ks}}.$$

Using Eq. (78), this derivative can be written for the fractional powers in the form

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \sum_{k,s=1}^{n,m} G_s^k \frac{\partial}{\partial q_{ks}^\alpha} + A \sum_{k,s=1}^{n,m} F_s^k \frac{\partial}{\partial p_{ks}^\alpha}. \tag{81}$$

The α-omega function is defined by the equation

$$\Omega_\alpha = \sum_{k,s=1}^{n,m} (\{G_s^k, p_{ks}^\alpha\}_\alpha + A\{q_{ks}^\alpha, F_s^k\}_\alpha). \tag{82}$$

Here we use the following notations for the fractional Poisson brackets:

$$\{A, B\}_\alpha = \sum_{k,s=1}^{n,m} \left(\frac{\partial A}{\partial q_{ks}^\alpha} \frac{\partial B}{\partial p_{ks}^\alpha} - \frac{\partial A}{\partial p_{ks}^\alpha} \frac{\partial B}{\partial q_{ks}^\alpha} \right). \tag{83}$$

Using Eqs. (79), (82), and (81), we get the Liouville equation in the form

$$\frac{\partial \widetilde{\rho}_n}{\partial t} = \Lambda_n \widetilde{\rho}_n, \tag{84}$$

where Λ_n is Liouville operator that is defined by the equation

$$\Lambda_n \widetilde{\rho}_n = - \sum_{k,s=1}^{n,m} \left(\frac{\partial (G_s^k \widetilde{\rho}_n)}{\partial q_{ks}^\alpha} + A \frac{\partial (F_s^k \widetilde{\rho}_n)}{\partial p_{ks}^\alpha} \right). \tag{85}$$

B. First fractional Bogoliubov equation

The Bogoliubov equations [11–14] are equations for the reduced distribution functions. These equations can be derived from the Liouville equation. Let us derive the first fractional Bogoliubov equation (93) from the fractional Liouville equation (84).

In order to derive the equation for the function $\widetilde{\rho}_1$ we differentiate Eq. (75), which defines one-particle reduced distribution:

$$\frac{\partial \widetilde{\rho}_1}{\partial t} = \frac{\partial}{\partial t} \hat{I}^\alpha[2, \ldots, n] \widetilde{\rho}_n = \hat{I}^\alpha[2, \ldots, n] \frac{\partial \widetilde{\rho}_n}{\partial t}.$$

Using the Liouville equation (84) for n-particle distribution function $\widetilde{\rho}_n$, we have

$$\frac{\partial \widetilde{\rho}_1}{\partial t} = \hat{I}^\alpha[2, \ldots, n] \Lambda_n \widetilde{\rho}_n(\mathbf{q}, \mathbf{p}, t). \tag{86}$$

Substituting Eq. (85) in Eq. (86), we get

$$\frac{\partial \widetilde{\rho}_1}{\partial t} = - \hat{I}^\alpha[2, \ldots, n] \sum_{k,s=1}^{n,m} \left(\frac{\partial (G_s^k \widetilde{\rho}_n)}{\partial q_{ks}^\alpha} + A \frac{\partial (F_s^k \widetilde{\rho}_n)}{\partial p_{ks}^\alpha} \right). \tag{87}$$

Let us consider in Eq. (87) the integration over q_{ks} and p_{ks} for the k-particle term. Since the coordinates and momenta are independent variables, we can derive

$$\hat{I}^\alpha[q_{ks}]\frac{\partial}{\partial q_{ks}^\alpha}(G_s^k\widetilde{\rho}_n) = \frac{1}{\alpha\Gamma(\alpha)}(G_s^k\widetilde{\rho}_n)_{-\infty}^{+\infty} = 0. \qquad (88)$$

Here we use the condition

$$\lim_{q_{ks}\to\pm\infty}\widetilde{\rho}_n = 0, \qquad (89)$$

which follows from the normalization condition. If limit (89) is not equal to zero, then the integration over phase space is equal to infinity. Similarly, we have

$$\hat{I}^\alpha[p_{ks}]\left(\frac{\partial}{\partial p_{ks}^\alpha}(F_s^k\widetilde{\rho}_n)\right) = \frac{1}{\alpha\Gamma(\alpha)}(F_s^k\widetilde{\rho}_n)_{-\infty}^{+\infty} = 0.$$

Then all terms in Eq. (87) with $k=2,\ldots,n$ are equal to zero. We have only term for $k=1$. Therefore Eq. (87) has the form

$$\frac{\partial\widetilde{\rho}_1}{\partial t} = -\sum_s^m \hat{I}^\alpha[2,\ldots,n]\left(\frac{\partial(G_s^1\widetilde{\rho}_n)}{\partial q_{1s}^\alpha} + A\frac{\partial(F_s^1\widetilde{\rho}_n)}{\partial p_{1s}^\alpha}\right). \qquad (90)$$

Since the variable $\mathbf{q}_1$ is an independent of $\mathbf{q}_2,\ldots\mathbf{q}_n$ and $\mathbf{p}_2,\ldots\mathbf{p}_n$, the first term in Eq. (90) can be written in the form

$$\sum_{s=1}^m \hat{I}^\alpha[2,\ldots,n]\frac{\partial(G_s^1\widetilde{\rho}_n)}{\partial q_{1s}^\alpha} = \sum_{s=1}^m \frac{\partial}{\partial q_{1s}^\alpha}G_s^1\hat{I}^\alpha[2,\ldots,n]\widetilde{\rho}_n$$

$$= \sum_{s=1}^m \frac{\partial(G_s^1\widetilde{\rho}_1)}{\partial q_{1s}^\alpha}.$$

The force F_s^1 acts on the first particle. This force is a sum of the internal forces $F_s^{1k}=F_s(\mathbf{q}_1,\mathbf{p}_1,\mathbf{q}_k,\mathbf{p}_k,t)$, and the external force $F_s^{1e}=F_s^e(\mathbf{q}_1,\mathbf{p}_1,t)$. In the case of binary interactions, we have

$$F_s^1 = F_s^{1e} + \sum_{k=2}^n F_s^{1k}. \qquad (91)$$

Using Eq. (91), the second term in Eq. (90) can be rewritten in the form

$$\hat{I}^\alpha[2,\ldots,n]\left(\frac{\partial(F_s^1\widetilde{\rho}_n)}{\partial p_{1s}^\alpha}\right)$$

$$= \hat{I}^\alpha[2,\ldots,n]\left(\frac{\partial(F_s^{1e}\widetilde{\rho}_n)}{\partial p_{1s}^\alpha} + \sum_{k=2}^n \frac{\partial(F_s^{1k}\widetilde{\rho}_n)}{\partial p_{1s}^\alpha}\right)$$

$$= \frac{\partial(F_s^{1e}\widetilde{\rho}_1)}{\partial p_{1s}^\alpha} + \sum_{k=2}^n \frac{\partial}{\partial p_{1s}^\alpha}\hat{I}^\alpha[2,\ldots,n](F_s^{1k}\widetilde{\rho}_n). \qquad (92)$$

Since the n-particle distribution function $\widetilde{\rho}_n$ is a symmetric function for the identical particles, we have that all $(n-1)$ terms of sum (92) are identical. Therefore the sum can be replaced by one term with the multiplier $(n-1)$:

$$\sum_{k=2}^n \hat{I}^\alpha[2,\ldots,n]\frac{\partial}{\partial p_{1s}^\alpha}(F_s^{1k}\widetilde{\rho}_n) = (n-1)\hat{I}^\alpha[2,\ldots,n]\frac{\partial}{\partial p_{1s}^\alpha}(F_s^{12}\widetilde{\rho}_n).$$

Using $\hat{I}^\alpha[2,\ldots,n]=\hat{I}^\alpha[2]\hat{I}^\alpha[3,\ldots,n]$, we have

$$\hat{I}^\alpha[2]\frac{\partial}{\partial p_{1s}^\alpha}(F_s^{12}\hat{I}^\alpha[3,\ldots,n]\widetilde{\rho}_n) = \frac{\partial}{\partial p_{1s}^\alpha}\hat{I}^\alpha[2](F_s^{12}\widetilde{\rho}_2).$$

Here we use definition (77) of two-particle distribution function. Since $\mathbf{p}_1$ is independent of $\mathbf{q}_2$, $\mathbf{p}_2$, we can change the order of the integrations and the differentiations:

$$\hat{I}^\alpha[2]\frac{\partial}{\partial p_{1s}^\alpha}(F_s^{12}\widetilde{\rho}_2) = \frac{\partial}{\partial p_{1s}^\alpha}\hat{I}^\alpha[2]F_s^{12}\widetilde{\rho}_2.$$

Finally, we obtain the equation for one-particle reduced distribution function,

$$\frac{\partial\widetilde{\rho}_1}{\partial t} + \sum_{s=1}^m \frac{\partial(G_s^1\widetilde{\rho}_1)}{\partial q_{1s}^\alpha} + A\sum_{s=1}^m \frac{\partial(F_s^{1e}\widetilde{\rho}_1)}{\partial p_{1s}^\alpha} = (n-1)AI(\widetilde{\rho}_2). \qquad (93)$$

Here $I(\widetilde{\rho}_2)$ is a term with two-particle reduced distribution function,

$$I(\widetilde{\rho}_2) = -\sum_{s=1}^m \frac{\partial}{\partial p_{1s}^\alpha}\hat{I}^\alpha[2]F_s^{12}\widetilde{\rho}_2. \qquad (94)$$

Equation (93) is called a *first fractional Bogoliubov equation* (first equation of Bogoliubov chain).

Let us consider the physical meaning of the term $I(\widetilde{\rho}_2)$. The term $I(\widetilde{\rho}_2)d\mu_\alpha(\mathbf{q},\mathbf{p})$ is a velocity of particle number change in $4m$-dimensional elementary phase volume $d\mu_\alpha(\mathbf{q}_1,\mathbf{p}_2,\mathbf{q}_2,\mathbf{p}_2)$. This change is caused by the interactions between particles. If $\alpha=1$, then we have the first Bogoliubov equation for non-Hamiltonian systems.

C. Second fractional Bogoliubov equation

The fractional Liouville equation allows us to derive equation for two-particle reduced distribution function $\widetilde{\rho}_2$ in the form

$$\frac{\partial\widetilde{\rho}_2}{\partial t} = \sum_{k=1}^2 \Lambda_k\widetilde{\rho}_2 + \Lambda_{12}\widetilde{\rho}_2 + c(n)AI(\widetilde{\rho}_3), \qquad (95)$$

where $c(n)=(n-1)(n-2)/2$, and Λ_k is one-particle Liouville operator,

$$\Lambda_k\widetilde{\rho}_2 = -\sum_{s=1}^m \frac{\partial(G_s^k\widetilde{\rho}_2)}{\partial q_{ks}^\alpha} - A\sum_{s=1}^m \frac{\partial(F_s^{ke}\widetilde{\rho}_2)}{\partial p_{ks}^\alpha},$$

and Λ_{12} is two-particle Liouville operator,

$$\Lambda_{12}\widetilde{\rho}_2 = A\sum_{s=1}^m \frac{\partial}{\partial p_{1s}^\alpha}(F_s^{12}\widetilde{\rho}_2) + A\sum_s \frac{\partial}{\partial p_{2s}^\alpha}(F_s^{21}\widetilde{\rho}_2),$$

and $I(\widetilde{\rho}_3)$ is a term with the three-particle reduced distribution,

$$I(\widetilde{\rho}_3) = \sum_{s=1}^m \hat{I}^\alpha[3]\left(\frac{\partial(F_s^{13}\widetilde{\rho}_3)}{\partial p_{1s}^\alpha} + \frac{\partial(F_s^{23}\widetilde{\rho}_3)}{\partial p_{2s}^\alpha}\right). \qquad (96)$$

The derivation of Eq. (95) is the analogous to the derivation of Eq. (93).

It is easy to see that Eqs. (93) and (95) are not closed. The system of equations for the reduced distribution functions are called the Bogoliubov hierarchy equations.

VI. FRACTIONAL VLASOV EQUATION AND DEBYE RADIUS

A. Fractional Vlasov equation

In this section, we derive the fractional analog of the Vlasov equation from the first fractional Bogoliubov equation. Let us consider the particles as statistical independent systems. In this case, we have

$$\widetilde{\rho}(\mathbf{q}_1,\mathbf{p}_1,\mathbf{q}_2,\mathbf{p}_2,t) = \widetilde{\rho}(\mathbf{q}_1,\mathbf{p}_1,t)\widetilde{\rho}(\mathbf{q}_2,\mathbf{p}_2,t). \quad (97)$$

Substituting Eq. (97) in Eq. (94), we get

$$I(\widetilde{\rho}_2) = -\sum_s^m \frac{\partial}{\partial p_{1s}^\alpha} \widetilde{\rho}_1[1] \hat{I}^\alpha[2] F_s^{12} \widetilde{\rho}_1[2].$$

Here we use the notation $\rho[k]$ for the distribution function $\rho(\mathbf{q}_k,\mathbf{p}_k,t)$. As the result we have the effective forces,

$$F_s^{eff}(\mathbf{q}_1,\mathbf{p}_1,t) = \hat{I}^\alpha[2] F_s^{12} \widetilde{\rho}_1[2].$$

In this case, we can rewrite the term (94) in the form

$$I(\widetilde{\rho}_2) = -\frac{\partial}{\partial p_{1s}^\alpha} (\widetilde{\rho}_1 F_s^{1eff}). \quad (98)$$

Substituting Eq. (98) in Eq. (93), we get

$$\frac{\partial \widetilde{\rho}_1}{\partial t} + \sum_{s=1}^m \frac{\partial (G_s^1 \widetilde{\rho}_1)}{\partial q_{1s}^\alpha} + A\sum_{s=1}^m \frac{\partial}{\partial p_{1s}^\alpha} [(F_s^{1e} + bF_s^{1eff}) \widetilde{\rho}_1] = 0, \quad (99)$$

where $b=n-1$. This equation is a closed equation for one-particle distribution function with the external force F^{1e} and the effective force F^{1eff}. Equation (99) can be called a fractional Vlasov equation [37].

The fractional Liouville, Bogoliubov, and Vlasov equation are a better approximation than its classical analogs for the systems with the fractional phase spaces (the fractal dimensional spaces). For example, the systems that live on some fractals (spaces with the fractal dimensions) can be described by the suggested fractional equations.

B. Debye radius

In this section, we consider the Debye radius for the fractional systems that are defined by the equations

$$\frac{dq_{ks}^\alpha}{dt} = \frac{p_{ks}^\beta}{m}, \quad \frac{dp_{ks}^\alpha}{dt} = AF_s^k(q,p), \quad (100)$$

where we use the dimensionless variables q_k, p_k, F_k, t. Let $r_0 = q_0$ be the radius of the interaction. Here $m = Mr_0/t_0 p_0$ is a dimensionless mass, where M is a particle mass. Using $m = 1$, we get

$$t_0 = \frac{Mq_0}{p_0}, \quad A = \frac{t_0 F_0}{p_0} = \frac{Mq_0}{p_0^2}. \quad (101)$$

Let us use the condition $p_0^2/M = kT_0$ for the characteristic momentum p_0. Note that the condition $p^2/M = kT$ can be realized for non-Hamiltonian and dissipative systems [21].

The first fractional Bogoliubov equation (93) for the dimensionless one-particle distribution $\widetilde{\rho}_1$ has the following dimensionless form [38]:

$$\frac{\partial \widetilde{\rho}_1}{\partial t} + \sum_{s=1}^3 \frac{\partial (p_{1s}^\beta \widetilde{\rho}_1)}{\partial q_{1s}^\alpha} + A\sum_{s=1}^3 \frac{\partial (F_s^1 \widetilde{\rho}_1)}{\partial p_{1s}^\alpha} = ABI(\widetilde{\rho}_2). \quad (102)$$

The dimensionless first Bogoliubov equation (102) has two characteristic parameters,

$$A = \frac{t_0 F_0}{p_0} = \frac{Mr_0 F_0}{p_0^2} = \frac{r_0 F_0}{kT_0}, \quad B = n_0 r_0^{3\alpha}. \quad (103)$$

Let us consider the coefficient B. It is known [18,19] that fractal particle system and fractal media are described by the power law relation (29):

$$n(r) = n_0 r^D, \quad (104)$$

where $D<3$ and n_0 is the D-dimensional concentration of the D-dimensional distribution of particles. The dimension D of fractal system is connected with order of the fractional integrals α by $D=3\alpha$. The concentration n_0 can be defined by the D-dimensional mass density k: $n_0 = k/M$ which is used in Eqs. (26) and (29). To calculate the mass fractal dimension D and concentration n_0, we can take the logarithm of both sides of Eq. (104). When we graph $\ln(n)$ as a function of $\ln(r)$, we have

$$\ln(n) = D\ln(r) + \ln(n_0),$$

and we get a value of the fractal dimension D of fractal media and parameter n_0. Therefore these values can be measured for homogeneous fractal media.

Let us consider the fractional systems with the force

$$F_{kl} = \frac{e^2}{4\pi\varepsilon_0 r_0^2} \frac{1}{|r_k - r_l|^{2\delta}}, \quad (105)$$

where r_k and r_l are dimensional values of coordinates. If $\delta = 1$, then we have the usual electrostatic interaction. In this case, the Gauss theorem for the fractional space is not satisfied. If $2\delta = 3\alpha - 1$, then the Gauss theorem for the fractional space is satisfied. The radius r_0 and the force F_0 are connected by the equation

$$F_0 = \frac{e^2}{4\pi\varepsilon_0} \frac{1}{r_0^2}. \quad (106)$$

Using the relation $AB \sim 1$, we have the characteristic radius of the interaction in the fractal media,

$$r_0 = r_D = \sqrt[3\alpha-1]{\frac{\varepsilon_0 kT_0}{e^2 n_0}}, \quad (107)$$

which can be called a fractional Debye radius. If the particle systems or media have the integer mass dimension $D=3$, then $\alpha=1$, and we get the usual equation for the Debye radius [39]. The fractional radius characterizes the scale $q_0 = r_0$ of the fractal media or fractal system with noninteger mass dimension.

VII. CONCLUSION

In this paper, we consider the fractional generalizations of the phase volume, the phase volume element, and the Poisson brackets. These generalizations lead us to the fractional analog of the phase space. The space can be considered as a fractal dimensional space. We consider systems on the fractional phase space and the fractional analogs of the Hamilton equations. The physical interpretation of the fractional phase space is discussed. The fractional generalization of the average values is derived.

It is known that the fractional derivative of a constant need not be zero. This relation leads to the correlation between coordinates q and momentum p. Therefore q and p are not independent variables in the usual sense. As the result, the generalization of the Poisson brackets with fractional derivatives (15) is not canonical. In order to derive equations with fractional derivatives we must have a generalization of Darboux theorem [40] for symplectic form with fractional exterior derivatives. However, this generalization is an open question at this moment. In order to define Poisson brackets with the usual relations for the coordinates and the momenta we can use Poisson brackets (22) with the fractional power of coordinates and momenta.

Note that the dissipative and non-Hamiltonian systems can have stationary states of the Hamiltonian systems [41]. Classical dissipative and non-Hamiltonian systems can have

the canonical Gibbs distribution as a solution of the stationary Liouville equations for this dissipative system [21]. Using the methods [21], it is easy to prove that some fractional dissipative systems can have fractional analog of the Gibbs distribution (non-Gaussian statistic) as a solution of the fractional Liouville equations. Using the methods [21], it is easy to find the stationary solutions of the fractional Bogoliubov equations for the fractional systems.

Note that the quantization of the fractional systems is a quantization of non-Hamiltonian dissipative systems. Using the method, which is suggested in Refs. [42–44], we can realize the Weyl quantization for the fractional systems. The suggested fractional Hamilton and Liouville equations allow us to derive the fractional generalization for the quantum systems by methods suggested in Refs. [42–44].

In this paper the fractional analogs of the Bogoliubov hierarchy equations are derived. In order to derive this analog we use the fractional Liouville equation [10], we define the fractional average values and the fractional reduced distribution functions. The fractional analog of the Vlasov equation and the Debye radius are considered.

The fractional Bogoliubov hierarchy equation can be used to derive the Enskog transport equation. The fractional analog of the hydrodynamics equations can be derived from the first fractional Bogoliubov equation. These equations will be considered in the next paper.

It is known that the Fokker-Planck equation can be derived from the Bogoliubov hierarchy equations [8]. The fractional Fokker-Planck equation can be derived from the fractional Bogoliubov equation. However, this fractional Fokker-Planck equation can be differed from the equation known in the literature [5–7].

The quantum generalization of the suggested fractional Bogoliubov equation can be considered by the methods that are suggested in Refs. [42–44].

[1] T. F. Nonnenmacher, J. Phys. A **23**, L697 (1990).
[2] W. G. Glockle and T. F. Nonnenmacher, J. Stat. Phys. **71**, 741 (1993).
[3] R. Metzler, W. G. Glockle, and T. F. Nonnenmacher, Physica A **211**, 13 (1994).
[4] M. F. Schlesinger, J. Stat. Phys. **36**, 639 (1984).
[5] R. Metzler and J. Klafter, Phys. Rep. **339**, 1 (2000).
[6] G. M. Zaslavsky, Phys. Rep. **371**, 461 (2002).
[7] G. M. Zaslavsky, Physica D **76**, 110 (1994).
[8] A. Isihara, *Statistical Physics* (Academic, New York, 1971), Appendix IV and Sec. 7.5.
[9] I. P. Kornfeld, Ja. G. Sinai, and S. V. Fomin, *Ergodic Theory* (Nauka, Moscow, 1980), Sec. 2.2.
[10] V. E. Tarasov, Chaos **14**, 123 (2004); e-print nlin.CD/0312044.
[11] N. N. Bogoliubov, *Selected Works* (Naukova Dumka, Kiev, 1970), Vol. 2.
[12] N. Bogolybow, J. Phys. (USSR) **10**, 265 (1946).
[13] K. P. Gurov, *Foundation of Kinetic Theory. Method of N. N. Bogoliubov* (Nauka, Moscow, 1966).
[14] D. Ya. Petrina, V. I. Gerasimenko, and P. V. Malishev, *Mathematical Basis of Classical Statistical Mechanics* (Naukova Dumka, Kiev, 1985).
[15] K. Cottrill-Shepherd and M. Naber, J. Math. Phys. **42**, 2203 (2001); e-print math-ph/0301013.
[16] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Integrals and Derivatives of Fractional Order and Applications* (Nauka i Tehnika, Minsk, 1987); *Fractional Integrals and Derivatives Theory and Applications* (Gordon & Breach, New York, 1993).
[17] J. C. Collins, *Renormalization* (Cambridge University Press, Cambridge, England, 1984), Sec. 4.1.
[18] B. Mandelbrot, *The Fractal Geometry of Nature* (Freeman, New York, 1983).
[19] M. Schroeder, *Fractals, Chaos, Power Laws* (Freeman, New York, 1990), Sec. 10.
[20] V. E. Tarasov, Theor. Math. Phys. **110**, 57 (1997).
[21] V. E. Tarasov, Mod. Phys. Lett. B **17**, 1219 (2003); e-print cond-mat/0311536.
[22] A. Sergi and M. Ferrario, Phys. Rev. E **64**, 056125 (2001).
[23] N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields* (Wiley, New York, 1980), Sec. 9.
[24] V. E. Tarasov, Mod. Phys. Lett. A **26**, 2411 (1994).

[25] V. E. Tarasov, Phys. Lett. B **323**, 296 (1994); e-print hep-th/0401223.

[26] M. E. Tuckerman, C. J. Mundy, and G. J. Martyna, Europhys. Lett. **45**, 149 (1999).

[27] M. E. Tuckerman, Y. Liu, G. Ciccotti, and G. J. Martyna, J. Chem. Phys. **115**, 1678 (2001).

[28] P. Minary, G. J. Martyna, and M. E. Tuckerman, J. Chem. Phys. **118**, 2510 (2003).

[29] J. D. Ramshaw, Europhys. Lett. **59**, 319 (2002).

[30] J. D. Ramshaw, Phys. Lett. A **116**, 110 (1986).

[31] A. Sergi, Phys. Rev. E **67**, 021101 (2003).

[32] A. Sergi, Phys. Rev. E **69**, 021109 (2004).

[33] G. S. Ezra, J. Math. Chem. **32**, 339 (2002).

[34] G. S. Ezra, J. Math. Chem. **35**, 29 (2004).

[35] V. E. Tarasov, Theor. Math. Phys. **110**, 168 (1997).

[36] N. N. Bogoliubov and N. N. Bogoliubov, Jr., *Introduction to Quantum Statistical Mechanics* (Nauka, Moscow, 1984), Sec. 1.4.

[37] A. A. Vlasov, Zh. Eksp. Teor. Fiz. **8**, 231 (1938).

[38] G. Sandri, Ann. Phys. (N.Y.) **24**, 332 (1963).

[39] P. Debye and E. Huckle, Phys. Z. **24**, 185, 305 (1923).

[40] B. A. Dubrovin, S. P. Novikov, and A. T. Fomenko, *Modern Geometry: Methods and Applications*, 2nd ed. (Nauka, Moscow, 1986), Secs. 26, 34.

[41] V. E. Tarasov, Phys. Rev. E **66**, 056116 (2002).

[42] V. E. Tarasov, Phys. Lett. A **288**, 173 (2001); e-print quant-ph/0311159.

[43] V. E. Tarasov, Moscow Univ. Phys. Bull. **56**, 5 (2001).

[44] V. E. Tarasov, Theor. Phys. **2**, 150 (2002).